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Extending States on Preordered Semigroups and the Existence of Quasitraces on C^* -Algebras

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It is shown that every state on a subsemigroup of a scaled preordered semigroup extends to the whole semigroup, extending a theorem of Goodearl and Handelman. As an application, every state on $K_0(A)$ for a unital C^* -algebra A comes from a quasitrace. © 1992 Academic Press, Inc.

1. INTRODUCTION

Ordered groups arise naturally in many contexts. One of the most fruitful ways to study ordered groups is via the states on the group; in fact, the order structure in some cases can be recovered from the state space [Bl1, 6.8.5]. One of the most important results in the theory of ordered groups is the Goodearl–Handelman theorem on extension of states from subgroups [GH].

One major application of the theory of ordered groups is in K -theory (see [Bl1, Sect. 6].) If A is a [simple] unital C^* -algebra, the structure of the ordered group $K_0(A)$ is of great interest, and in particular describing the states on $K_0(A)$ is very important. The states which can be easily described and studied are the ones coming from quasitraces on A ; it was shown in [Rø] that every state on $K_0(A)$ comes from a quasitrace if A is simple. See [Bl2] for a more complete discussion.

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It turns out that the problem of whether every state on $K_0(A)$ comes from a quasitrace can be rephrased as a state extension problem for ordered semigroups. And the analogous problem for nonsimple C^* -algebras requires the use of preordered semigroups. Ordered semigroups have already been of interest in C^* -algebra theory [Bl3], as well as in other contexts.

In this note, we generalize the Goodearl–Handelman theorem to preordered semigroups. As an application, we obtain an (almost) purely semigroup-theoretic proof of a generalization of the principal result of [Rø] to the nonsimple case. We have stated the semigroup results in the greatest possible generality, more than is needed for the applications. Extension results for states were established for semigroups equipped with the *algebraic* preordering by Tarski [Ta], and for preordered semigroups with cancellation (i.e., $x + z = y + z$ implies $x = y$) by Aumann [Au]. Preordered groups were considered in [Hd1]. For our applications we must deal with preordered semigroups which are not algebraically ordered and which do not have cancellation.

2. STATES ON PREORDERED SEMIGROUPS

DEFINITION 2.1. A *preordered semigroup* is an abelian semigroup S with a preorder (transitive relation) $<$ which is translation-invariant, i.e., if $x < y$, then $x + z < y + z$ for all $z \in S$. As usual, we write $x \leq y$ if $x < y$ or $x = y$; \leq is then a reflexive transitive relation.

An *order unit* in a preordered semigroup S is an element u such that, for every $x \in S$, there is an $n \in \mathbb{N}$ such that $x \leq x + u$, $x \leq nu$, and $u \leq x + nu$. Note that if u is an order unit and $u \leq v$, then v is also an order unit.

A *scaled preordered semigroup* is a preordered semigroup with distinguished order unit.

A *state* on a scaled preordered semigroup $(S, <, u)$ is an order-preserving homomorphism f from S to the additive group of real numbers \mathbb{R} , such that $f(u) = 1$. Denote the set of all states on $(S, <, u)$ by $\Delta(S, <, u)$, or just by $\Delta(S)$ when the order structure and order unit are understood. $\Delta(S)$ is a compact convex set in the topology of pointwise convergence (if $x \leq nu$ and $u \leq x + nu$, then $1 - n \leq f(x) \leq n$ for any $f \in \Delta(S)$.)

DEFINITION 2.2. Let ϕ be a homomorphism from a scaled preordered semigroup $(S, <_S, u)$ to a scaled preordered semigroup $(T, <_T, v)$, with $\phi(u) = v$.

ϕ is an *order embedding* if, for all $x, y \in S$, $x <_S y$ if and only if $\phi(x) <_T \phi(y)$.

ϕ is an *approximate order embedding* if, for any $x, y \in S$ with y and $\phi(y)$

order units, there is an $n \in \mathbb{N}$ with $nx + u \leq_S ny$ if and only if there is an $m \in \mathbb{N}$ with $m\phi(x) + v \leq_T m\phi(y)$.

ϕ is a *stable order embedding* if, for any $x, y \in S$, there are $n \in \mathbb{N}$ and $z \in S$ with $nx + z + u \leq_S ny + z$ if and only if there are $m \in \mathbb{N}$ and $w \in T$ with $m\phi(x) + w + v \leq_T m\phi(y) + w$.

It is obvious that an order embedding is an approximate order embedding. It is not *a priori* obvious that an order embedding is a stable order embedding, but this will follow from Proposition 2.4 below. Note that if g is a state on T and $\phi: S \rightarrow T$ is a stable order embedding, then $g \circ \phi$ is a state on S : if $x \leq y$ in S and $k \in \mathbb{N}$, then $kx + z + u \leq (ky + u) + z$ for any $z \in S$, and so $mk\phi(x) + w + v \leq mk\phi(y) + mv + w$ for some $w \in T$ and $m \in \mathbb{N}$. Thus $(g \circ \phi)(x) + 1/mk \leq (g \circ \phi)(y) + 1/k$, so $g \circ \phi$ is order-preserving.

LEMMA 2.3. *Let $(S, <, u)$ be a scaled preordered semigroup, and $x, y \in S$, with y an order unit. If there is a $z \in S$ such that $x + z + u \leq y + z$, then there is an $n \in \mathbb{N}$ with $nx + u \leq ny$.*

Proof. Note that

$$\begin{aligned} 2(x + u) + z &= (x + u) + (x + z + u) \leq (x + u) + (y + z) \\ &= y + (x + z + u) \leq y + y + z = 2y + z \end{aligned}$$

and by induction $mx + mu + z \leq my + z$ for all m . Choose k so that $z \leq ky$, and then choose m so that $kx + u \leq mu + z$. Then

$$\begin{aligned} (m + k)x + u &= mx + kx + u \leq mx + mu + z \leq my + z \leq my + ky \\ &= (m + k)y. \quad \blacksquare \end{aligned}$$

PROPOSITION 2.4. *Let $(S, <_S, u)$ and $(T, <_T, v)$ be scaled preordered semigroups, and ϕ a homomorphism from S to T with $\phi(u) = v$. Then ϕ is an approximate order embedding if and only if ϕ is a stable order embedding.*

Proof. Suppose ϕ is an approximate order embedding, $x, y \in S$, and $m\phi(x) + w + v \leq_T m\phi(y) + w$ for some $m \in \mathbb{N}$ and $w \in T$. Choose k large enough that $y + ku$ is an order unit in S and $\phi(y) + kv = \phi(y + ku)$ is an order unit in T . Then $m\phi(x + ku) + w + v \leq_T m\phi(y + ku) + w$, so by Lemma 2.3 there is an $r \in \mathbb{N}$ with $r\phi(x + ku) + v \leq_T r\phi(y + ku)$. Since ϕ is an approximate order embedding, there is an $n \in \mathbb{N}$ with $n(x + ku) + u \leq_S n(y + ku)$. Set $z = nku$. The other half of the proof that ϕ is a stable order embedding is similar.

Now suppose ϕ is a stable order embedding, and $x, y \in S$, with y an order unit, for which there is an $m \in \mathbb{N}$ with $m\phi(x) + v \leq_T m\phi(y)$. Then since ϕ is a stable order embedding there is a $k \in \mathbb{N}$ and $z \in S$

with $kx + z + u \leq_S ky + z$, so by Lemma 2.3 there is an $n \in \mathbb{N}$ with $nkx + u \leq_S nky$. The other half of the proof that ϕ is an approximate order embedding is similar. ■

A result similar to 2.4 for partially ordered groups was established by Handelman [Hd2, Lemma 5].

Remark 2.5. If ϕ is as in Definition 2.2, we say ϕ is a *weak stable order embedding* if, whenever $x, y \in S$ and $\phi(x) + v <_T \phi(y)$, there is an $n \in \mathbb{N}$ and $z \in S$ such that $nx + z + u <_S ny + z$, and whenever $x, y \in S$ and $x + u <_S y$, there is an $m \in \mathbb{N}$ and $w \in T$ such that $m\phi(x) + w + v <_T m\phi(y) + w$. It is obvious that a stable order embedding is a weak stable order embedding, and a proof similar to that of Proposition 2.4, using Lemma 2.3, shows that a weak stable order embedding is a stable order embedding.

Our main results are:

THEOREM 2.6. *Let $(S, <_S, u)$ and $(T, <_T, v)$ be scaled preordered semigroups, and let $\phi: S \rightarrow T$ be a homomorphism with $\phi(u) = v$. Then $\Delta(S) = \{g \circ \phi: g \in \Delta(T)\}$ if and only if ϕ is a stable order embedding.*

COROLLARY 2.7. *Let $(S, <, u)$ be a scaled preordered semigroup, S_0 a subsemigroup containing u . Give S_0 the relative preordering. Then every state on S_0 extends to S .*

Proof of 2.7. The inclusion map from S_0 into S is clearly an approximate order embedding, hence a stable order embedding by Proposition 2.4. The result then follows from Theorem 2.6. ■

Our approach is to obtain Corollary 2.7 as a corollary (special case) of Theorem 2.6. It may interest the reader to see the following argument for deducing Theorem 2.6 from Corollary 2.7. Suppose ϕ is a stable order embedding. Let $f \in \Delta(S)$, and argue as above Lemma 2.3 that $\phi(x) \leq \phi(y)$ implies $f(x) \leq f(y)$ for $x, y \in S$. Hence f induces $g \in \Delta(\phi(S))$ such that $g \circ \phi = f$; g extends to T by Corollary 2.7. However, this argument does not seem to simplify the proof of Theorem 2.6, since the best direct proof we know for Corollary 2.7 is essentially the same as our proof of Theorem 2.6.

Theorem 2.6 will be proved using the Goodearl–Handelman theorem, although it does not follow immediately. To explain the difficulties, we introduce some notation which will also be used in the proof of Theorem 2.6.

We first replace the scaled preordered semigroup $(S, <, u)$ by its maximal partially ordered quotient. Define an equivalence relation \sim on S by $x \sim y$ if $x \leq y$ and $y \leq x$, and denote the equivalence class of x by $\langle x \rangle$. Let $\bar{S} = \{\langle x \rangle: x \in S\}$. Define $\langle x \rangle + \langle y \rangle = \langle x + y \rangle$ and $\langle x \rangle \leq \langle y \rangle$ if $x \leq y$;

then one obtains a well-defined binary operation making \bar{S} into a partially ordered semigroup. Scale \bar{S} by taking $\langle u \rangle$ as order unit. There is an obvious one-one correspondence between the states on $(S, <, u)$ and the states on $(\bar{S}, \leq, \langle u \rangle)$. Going one step further, we may form the Grothendieck group $G(\bar{S})$; $G(\bar{S})$ becomes an ordered group in the sense of [GH] by taking $G(\bar{S})_+ = \{[x] - [y] : \langle x \rangle, \langle y \rangle \in \bar{S}, \langle y \rangle \leq \langle x \rangle\}$. (We denote the image of $\langle x \rangle$ in $G(\bar{S})$ by $[x]$.) The states of $(S, <, u)$ again correspond naturally to the states on $(G(\bar{S}), \leq, [u])$. Note that $[x] \leq [y]$ in $G(\bar{S})$ if and only if there is a $z \in S$ with $x + z \leq y + z$.

Even in the special case of Corollary 2.7 when S is partially ordered, the result does not follow immediately from the Goodearl–Handelman theorem for two technical reasons:

- (1) The natural map ϕ_G from $G(S_0)$ to $G(S)$ need not be injective in general.
- (2) Even if ϕ_G is injective, it need not be an order-isomorphism onto its image if the order on S is not strictly well-behaved in the sense of [B13].

There is a third technical difficulty in the general case of Theorem 2.6, in that ϕ need not drop to a well-defined homomorphism from $G(\bar{S})$ to $G(\bar{T})$.

We will apply the Goodearl–Handelman theorem in the following form:

LEMMA 2.8. *Let $(S, <, u)$ be a scaled preordered semigroup, and $x, y \in S$. Then $f(x) < f(y)$ for every $f \in \Delta(S)$ if and only if there is an $n \in \mathbb{N}$ and $z \in S$ with $nx + z + u \leq ny + z$.*

Proof. It is clear that $nx + z + u \leq ny + z$ for some $z \in S$ and $n \in \mathbb{N}$ implies $f(x) < f(y)$ for all $f \in \Delta(S)$. Conversely, suppose $f(x) < f(y)$ for all $f \in \Delta(S)$. Using the bijection between $\Delta(S)$ and $\Delta(G(\bar{S}))$, we obtain that $f([y] - [x]) > 0$ for all $f \in \Delta(G(\bar{S}))$, and it follows from [Go, 4.12] that $k([y] - [x])$ is an order unit in $G(\bar{S})$ for some $k \in \mathbb{N}$. So there is an $n \in \mathbb{N}$ with $n([y] - [x]) \geq [u]$, or equivalently $[nx + u] \leq [y]$, and we conclude that $nx + u + z \leq ny + z$ for some $z \in S$. ■

We will also need the following fact, which follows immediately from [B12, 3.4.7] (itself a consequence of the Goodearl–Handelman theorem; see also [Go, 7.11]):

LEMMA 2.9. *Let $(S, <, u)$ be a scaled preordered semigroup. If K is a closed convex subset of $\Delta(S)$ such that, for every $x, y \in S$, $f(x) < f(y)$ for all $f \in K$ implies $f(x) < f(y)$ for all $f \in \Delta(S)$, then $K = \Delta(S)$.*

Proof of Theorem 2.6. Set $K = \{g \circ \phi : g \in \Delta(T)\}$. If $K = \Delta(S)$, then it follows immediately from Lemma 2.8 that ϕ must be a stable order

embedding. Conversely, suppose ϕ is a stable order embedding. Then $K \subseteq \mathcal{A}(S)$, and by Lemma 2.8 K satisfies the hypotheses of Lemma 2.9, so $K = \mathcal{A}(S)$. ■

3. APPLICATIONS TO C^* -ALGEBRAS

In this section, we apply the state extension theorem to show that every state on the preordered group $K_0(A)$ for a unital C^* -algebra A comes from a quasitrace on A .

We first recall some facts from [Cu, Hd1, Bl2, Rø] about quasitraces, dimension functions, and semigroups of positive operators and projections. Let A be a unital C^* -algebra, and let $x, y \in M_\infty(A)_+ (= \lim_{\leftarrow} M_n(A)_+)$. Write $x \leq y$ if $r_j y r_j^* \rightarrow x$ for some sequence (r_j) in $M_\infty(A)$. Then \leq is symmetric and transitive. Write $x \sim y$ if there is a sequence s_j such that $(s_j^* s_j)$ [respectively $(s_j s_j^*)$] is an approximate identity for the hereditary C^* -subalgebra of $M_\infty(A)$ generated by x [respectively y]. Then \sim is an equivalence relation; if $x \sim y$, then $x \leq y$ and $y \leq x$, but the converse is not true in general. Let $\langle x \rangle$ denote the equivalence class of x , and let $W(A)$ be the set of all equivalence classes. Equip $W(A)$ with the structure of a preordered semigroup by setting $\langle x \rangle + \langle y \rangle = \langle x + y \rangle$ whenever x and y are orthogonal, and $\langle x \rangle \leq \langle y \rangle$ if $x \leq y$. This preordering is not the algebraic preordering, and $W(A)$ does not have cancellation in general. Let $V(A)$ be the subsemigroup of $W(A)$ of all equivalence classes of projections in $M_\infty(A)$. If p and q are projections and $\langle p \rangle \leq \langle q \rangle$, then $p = v^* v$ and $q \geq v v^*$ for some v , and so $\langle q \rangle = \langle q - v v^* \rangle + \langle p \rangle$. So the preordering on $V(A)$ inherited from $W(A)$ is the algebraic preordering. Also, $\langle 1_A \rangle$ is an order unit for $V(A)$ and $W(A)$.

States on $W(A)$ are called *dimension functions*, and each such function d can be lifted to $M_\infty(A)_+$ [or to $M_\infty(A)$] by $D(x) = d(\langle x \rangle)$ [respectively $D(a) = d(\langle a^* a \rangle)$.] The resulting dimension function satisfies the conditions in [Cu, Sect. 2.3]. Recall that a (normalized) *quasitrace* on A is a monotone homogeneous function $\tau: M_\infty(A)_+ \rightarrow \mathbf{R}_+$ satisfying $\tau(1_A) = 1$, $\tau(a^* a) = \tau(a a^*)$ for all $a \in M_\infty(A)$, and $\tau(x + y) = \tau(x) + \tau(y)$ if x and y commute. It is a well-known open problem whether all quasitraces are traces (i.e., additive everywhere.) Every quasitrace τ defines a lower semicontinuous dimension function D_τ by $D_\tau(x) = \lim_{n \rightarrow \infty} \tau(x^{1/n})$.

PROPOSITION 3.1. *If D is a dimension function on A , then there is a lower semicontinuous dimension function which agrees with D on projections ([Bl2, 6.4.3]; see [Rø, 4.1] for a proof.)*

PROPOSITION 3.2. *Every lower semicontinuous dimension function comes from a quasitrace [BH, II.2.2].*

The following result was proved in [Rø, 6.1] in the case where A is simple. If p is a projection in $M_\infty(A)$, denote by $[p]$ the class of p in $K_0(A)$.

THEOREM 3.3. *Let A be a unital C^* -algebra, and let $f: K_0(A) \rightarrow \mathbf{R}$ be a homomorphism such that $f([p]) \geq 0$ for all projections $p \in M_\infty(A)$, and $f([1_A]) = 1$. Then there is a quasitrace τ on A such that $f([p]) = \tau(p)$ for every projection $p \in M_\infty(A)$.*

Proof. If $p, q \in M_\infty(A)$ are projections such that $\langle p \rangle \leq \langle q \rangle$ in $V(A)$, then $p = v^*v$ and $q \geq vv^*$ for some $v \in M_\infty(A)$, whence $[q] = [p] + [q - vv^*]$ in $K_0(A)$ and so $f([p]) \leq f([q])$. Thus $\tilde{f}(\langle p \rangle) = f([p])$ defines a state on $V(A)$. Extend \tilde{f} to a state d on $W(A)$, let D be the corresponding dimension function, let \tilde{D} be a lower semicontinuous dimension function agreeing with D on projections, and let τ be the corresponding quasitrace. ■

If $(K_0(A), K_0(A)_+, [1_A])$ is regarded as a scaled preordered group in the standard way, then the functions satisfying the conditions of Theorem 3.3 are precisely the states on $K_0(A)$. $K_0(A)$ is (partially) ordered if A is stably finite [Bl1, 6.3.3]. A scaled ordered group always has a state by the Goodearl–Handelman theorem. Hence a unital C^* -algebra has a quasitrace if and only if it has a nontrivial stably finite quotient (cf. [Hd1].)

Very recently, Uffe Haagerup [Ha] has proved that every quasitrace on an exact C^* -algebra is a trace. This result applies to all nuclear and, more generally, to all subnuclear C^* -algebras. So we have

COROLLARY 3.4. *Let A be a unital exact C^* -algebra, and let $f: K_0(A) \rightarrow \mathbf{R}$ be a homomorphism such that $f([p]) \geq 0$ for all projections $p \in M_\infty(A)$, and $f([1_A]) = 1$. Then there is a tracial state τ on A such that $f([p]) = \tau(p)$ for every projection $p \in M_\infty(A)$.*

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